ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 129–152

MIXING EXAMPLES IN THE CLASS OF PIECEWISE MONOTONE AND CONTINUOUS MAPS OF THE UNIT INTERVAL

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ABSTRACT

A process (T,P) is said to have the "$\bar{d} > \delta$" property if there is a uniform, positive lowerbound δ on the $\bar{d}$ separation between the T–P names of (almost) every pair of points $x \neq y$. A finite group rotation with partition into distinct points provides a trivial example. Given any process having the $\bar{d} > \delta$ property we show that there exists a Bernoulli shift B so that $T \times B$ is measurably isomorphic to the natural extension of a piecewise monotone, continuous, and expanding map of the unit interval.

This construction is applied to produce interval maps which are ergodic but not weak-mixing, weak-mixing but not mixing, and mixing but not exact with respect to their unique absolutely continuous invariant measures, in contrast with the results known for piecewise $C^{1+\epsilon}$ expansive interval maps. In obtaining these examples we identify a number of non-trivial classes of automorphisms T which admit processes having the $\bar{d} > \delta$ property.

1

Let $I = [0,1]$ equipped with Lebesgue measure λ on the Lebesgue subsets $\mathcal{B}$. We shall be concerned with measurable point mappings $g: I \to I$ which satisfy:

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* Supported by NSERC grant OGP0046586 90.

Received May 24, 1990 and in revised form June 2, 1991

(1.1) There exists a finite or countably infinite collection of closed intervals $I_n \subseteq I$, $I_n = [a_n, b_n]$, such that $\lambda(I_n \cap I_m) = 0$ iff $m \neq n$ and $\bigcup_n I_n = I$, and such that

(1.2) Restricted to each I_n, $g|_{I_n}$ is continuous and monotone (non-increasing or non-decreasing).

We shall call such mappings **piecewise monotone and continuous** (p.m.c.). To study the dynamics of such a map one looks for an invariant measure μ for g; to avoid trivialities one specifies $\mu \ll \lambda$. This is an old idea going back at least to Renyi [R], Khinchine [Kh] and Doeblin [D], and the numerous articles cited there concerning number theoretic transformations.

The problem becomes tractable when we impose additional conditions on g. Here is a much studied situation. Let g be p.m.c and satisfy

(1.3) There exists a $\lambda \geq 1$ such that, for all n

$$\operatorname*{ess\,inf}_{x \in I_n}\{|g'(x)|\} \geq \lambda.$$

(1.4) $g|_{I_n}$ is twice continuously differentiable.

In case there are finitely many intervals I_n Lasota and Yorke [L Y] proved the existence of absolutely continuous invariant measures for g satisfying 1.1) – 1.4). Under the same conditions Bowen [Bow] went on to show that if g is weak-mixing with respect to such an invariant μ, then the natural extension of g (with respect to μ) is automatically a Bernoulli shift.

Earlier, Adler [Ad] had found unique invariant measures for p.m.c. g satisfying 1.3) and 1.4) with infinitely many intervals I_n but with the additional conditions

$$(1.5) \qquad \exists M, \quad \forall n \sup_{x,y,z \in I_n} \left| \frac{g''(x)}{g'(y)g'(z)} \right| \leq M.$$

(Trivial for finitely many intervals)

$$(1.6) \qquad g(I_n) = I.$$

1.5) is known as Renyi's condition. Bowen and Series [Bow S] observed that 1.6) may be weakened to the *Markov Condition*

$$(1.7) \qquad \text{If} \quad \Omega = \bigcup_n \{ \lim_{x \to a_n^+} g(x), \lim_{x \to b_n^+} g(x)\} \quad \text{then} \quad \Omega \subseteq \bigcup_n \{a_n, b_n\}$$

and, for all l, k

$$I_l \subseteq \bigcup_{n \geq 0} g^n(I_k).$$

Under conditions 1.1) – 1.5) and with 1.7) their result ensures a *unique* invariant measure $\mu \ll \lambda$ and it turns out that the natural extension of g with respect to μ will be a Bernoulli shift.

A number of authors have weakened Renyi's condition 1.5) obtaining analogues of the Bowen and Series theorem for not necessarily C^2 maps. See, for example, [W], [Ke], and [Bo2]. A common theme amongst these results is that conditions sufficient to ensure the existence of a unique absolutely continuous invariant measure imply, with respect to this measure, the natural extension of g is a Bernoulli shift.

The purpose of this article is to show that, at least in the class of p.m.c. maps with unique absolutely continuous invariant measures, this type of behavior (i.e., weak–mixing $\Rightarrow$ Bernoulli) is not inevitable. In particular, we construct p.m.c. maps g with Lebesgue measure the unique absolutely continuous invariant measure and which are, in order, ergodic but not weak-mixing, weak-mixing but not mixing and mixing but not exact (natural extension not a K–automorphism).

The central idea is contained in Section 3 where we describe a general method for extracting a p.m.c. interval map as a factor of the direct product of an abstract dynamical system (S, ν) and a Bernoulli shift. We require the system (S, ν) to possess the "$\bar{d} > \delta$ property": a lowerbounded $\bar{d}$ separation between (almost every) pair of names with respect to some generating partition. This property seems new to the literature and appears to be of interest beyond its use in our constructions. The bridge between the above dynamical system and the class of p.m.c. interval maps is provided by the generalized baker's transformation.

In Section 4 we construct our first two examples. Section 5 is reserved for the mixing, but not exact case, where it turns out any mixing rank 1 automorphism will suffice for the transformation S. In Section 2 we will establish notation, conventions, and sketch a few facts about rank 1 transformations we shall be using.

The results in this article have been greatly clarified through the helpful comments of D. Rudolph and A. del Junco. The referee has generously offered an alternate, more elegant proof of Proposition 5.3 which we present — both proofs are based on the work of Kalikow [Ka]. The article of King [Ki] is a good source

for background material on rank 1 transformations. We are pleased to acknowledge these contributions.

2

In this section we establish our notation and a few preliminary lemmas, wherever possible adhering to what is standard. The knowledgeable reader, particularly one familiar with the rank-1 block constructions, may prefer to begin directly with Section 3.

By a **dynamical system** $(X, \mathcal{B}\mu, T)$ we shall mean a Lebesgue probability space $(X, \mathcal{B}, \mu)$ equipped with a point mapping $T: X \to X$ which is measurable and measure–preserving: $T^{-1}A \in \mathcal{B}$ and $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

A **partition** $P = \{P_i\}_{i\in\Pi}$ will be a finite or countable collection of disjoint subsets $P_i \in \mathcal{B}$ satisfying $\mu(\bigcup P_i) = 1$. Partition elements P_i satisfying $\mu(P_i) > 0$ will be called **atoms of** P. Given $P = \{P_i\}_{i\in\Pi}$, and $Q = \{Q_j\}_{j\in\Pi'}$ of the same space we define the **join** $P \vee Q = \{P_i \cap Q_j\}_{\substack{i\in\Pi \\ j\in\Pi'}}$ or **common refinement** of P and Q. We say that P **refines** Q (and write $Q < P$) if each atom of Q is (up to null sets) a union of atoms of P. We say that P ϵ-**refines** Q (and write $Q <^{\epsilon} P$) if there exists a $\bar{Q} < P$ having the same number of atoms as Q and $\sum_i \mu(Q_i \Delta \tilde{Q}_i) < \epsilon$. In the presence of T, P gives rise to countably many partitions $T^k P = \{T^k P_i\}_{i\in\Pi}$ ($k \in \mathbb{Z}$ if T invertible, $k \in \mathbb{Z}^-$ if T is not.) We say that P **generates** $\mathcal{B}$ under T if the smallest σ-algebra containing all the $T^k P$ is $\mathcal{B}$. It is an important observation for us that, since X is a Lebesgue space, this statement is equivalent to requiring that the collection $\{T^k P\}$ separates (μ-almost all) points of X.

By a **process** we shall mean a pair consisting of a dynamical system $(X, \mathcal{B}, \mu, T)$ with T invertible and a partition P of X. We write $(X, \mathcal{B}, \mu, T, P)$ or when there is little danger of confusion about the underlying measure space, simply (T, P).

Given a process (T, P) and a point $x \in X$ we may consider the $T - P$ *name* of x, a member of $\Pi^{\mathbb{Z}}$, denoted $x_{-\infty}^{\infty}$ and defined by

$$(x_{-\infty}^{\infty})_j = i_0 \Leftrightarrow T^j x \in P_{i_0} \qquad (\Leftrightarrow x \in T^{-j} P_{i_0}).$$

Evidently, μ a.e. $x \in X$ has a well defined $T - P$ name. If $x_{-\infty}^{\infty} \in \Pi^{\mathbb{Z}}$ and $m \leq n$ we denote by $x_m^n = x_m x_{m+1} \dots x_n$. x_k means x_k^k.

If (T,P) is a process, $m \le 0 \le n$ integers and $i_m i_{m+1} \dots i_n \in \Pi^{n-m+1}$ a finite string of symbols we form the **elementary cylinder**

$$\{x \in X \mid x_m^n = i_m i_{m+1} \dots i_n\} = T^{-m} P_{i_m} \cap T^{-m-1} P_{i_{m+1}} \cdots \cap T^{-n} P_{i_n},$$

measurable w.r.t. $\bigvee_m^n T^{-i}P$. If T is ergodic, we may apply the Birkhoff Ergodic Theorem obtaining a subset $X' \subseteq X$, $\mu(X') = 1$ so that if $x' \in X'$ and C is an elementary cylinder, $C = \{x \mid x_m^n = i_m i_{m+1} \dots i_n\}$,

$$\mu(C) = \lim_{N\to\infty} \frac{1}{N-(n-m+1)} \{i \in [0,\ N-(n-m+1)) | x_i^{i+(n-m)} = i_m i_{m+1} \dots i_n\}.$$

We call such x' **generic** for (T,P).

Let A be a finite or countable set of distinct symbols and let $X = A^{\mathbf{Z}}$. Define $S:\ X \to X$ by $(Sx)_i = x_{i+1}$, the "shift to the left". Any S-invariant probability measure μ on the product (of discrete) σ-algebra gives rise to a **shift dynamical system.** Let $P_a = \{x \in A^{\mathbf{Z}} | x_0 = a\}$. Then $P = \{P_a\}_{a\in A}$ is called the **time–zero partition** and P is obviously a generator under S.

Shift systems are fundamental in the following sense. Let $(X, \mathcal{B}, \mu, T)$ be any dynamical system with T invertible and suppose $P = \{P_i\}_{i\in\Pi}$ generates under T. The $T-P$ name of $x \in X$ defines a 1–1 mapping from (μ a.e.) $x \in X$ to $x_{-\infty}^{\infty} \in \Pi^{\mathbf{Z}}$ which carries T to the shift S, and μ to a shift invariant measure on $\Pi^{\mathbf{Z}}$. The original dynamical system is therefore measurably isomorphic to this shift system. The partition P is sent to the time-zero partition of $\Pi^{\mathbf{Z}}$.

Let $\alpha \in A^N$ and $\beta \in A^M$ be two finite strings, say $\alpha = a_0 a_1 \dots a_{N-1}$ and $\beta = b_0 b_1 \dots b_{M-1}$. We define the **concatenation** of α and β, an element of A^{N+M} by

$$\alpha \otimes \beta = a_0 a_1 \dots a_{N-1} b_0 b_1 \dots b_{M-1}.$$

If $N = M$ then we measure the $\bar{d}$-**distance** between α and β by

$$\bar{d}(\alpha,\beta) = \frac{1}{N}\#\{i \in [0,N) \quad |a_i \ne b_i\}.$$

We denote by $|\alpha|$, $|\beta|$ (or sometimes $a = |\alpha|$, $b = |\beta|$) the lengths of α and β.

As usual, a **Rohlin stack of height** b is a partition $R = \{A_1, A_2, \dots, A_b, E\}$ with $TA_i = A_{i+1}$, $1 \le i \le b-1$. The set E is called the **error set**. A fundamental and useful observation in ergodic theory is that every aperiodic T

on a Lebesgue space X admits arbitrarily long Rohlin stacks with arbitrarily small E. Restricting this notion further gives a geometric (and possibly the most intuitive) definition of a rank 1 automorphism.

Definition 2.1: We say $T: X \to X$ is **rank 1** if there is a sequence of Rohlin stacks $R_1, R_2, \ldots$ so that for each $\epsilon > 0$ and finite partition Q of X, there exists an n so $\mu(E_n) < \epsilon$ and $Q \overset{\epsilon}{<} R_n$. ∎

Remarks 2.2:

(i) The condition $\mu(E_n) < \epsilon$ ensures that that Q is also well approximated (within 2ϵ) by unions of the *levels* of the n stack $\{A_1^{(n)}, A_2^{(n)}, \ldots, A_{b_n}^{(n)}\} = R_n - \{E_n\}$.

(ii) Suppose one finds a T-invariant set A with $0 < \mu(A) \leq \frac{1}{2}$. Choose a stack $R_n = \{A_1^{(n)}, A_2^{(n)}, \ldots, A_{b_n}^{(n)}, E_n\}$ whose levels approximate A very well; in particular, there exists a level $A_i^{(n)} \in R_n$ so $\mu(A \cap A_i^{(n)}) > \frac{4}{5}\mu(A_i^{(n)})$. But then this same inequality must be true for all $A_j^{(n)}$, $1 \leq j \leq b_n$ and we conclude that A has, instead, large measure. Thus any rank 1 T is automatically *ergodic*.

(iii) By (ii) we see that if T is rank 1 it is either periodic or aperiodic (*i.e.;* set of periodic points is of measure zero).

(iv) It will be convenient for us to always work with a subsequence of the R_n which we now describe. Let $Q_1 < Q_2 < \ldots$ be a generating sequence of finite partitions and let $\epsilon_n \downarrow 0$. For each k choose $R_{n_k} \overset{\epsilon_k}{>} Q_k$ with $\mu(E_{n_k}) < \epsilon_k$. Then, for all Q and $\epsilon > 0$, for all large enough k, $Q \overset{\epsilon}{<} R_{n_k}$ and $\mu(E_{n_k}) < \epsilon$. For aperiodic T, this assumption, combined with Rohlin's Lemma allows us to assume $\lim_k b_{n_k} = \infty$.

(v) Let $P = \{P_a\}_{a \in A}$ be a generator for T and let $S: A^{\mathbf{Z}} \to A^{\mathbf{Z}}$ be the related (isomorphic) shift automorphism. Let $\epsilon > 0$ be given. Then there is a single string $\beta = a_1 a_2 \ldots a_{b_n}$ of symbols from A so that for almost every $x \in A^{\mathbf{Z}}$ and for all large L we may make an "idealized" copy of x_0^L denoted $\tilde{x}_0^L$, a concatenation of copies of β interspersed with spacer symbols from A, the spacers occupying at most a fraction ϵ of the indices, so that

$$\bar{d}(x_0^L, \tilde{x}_0^L) < \epsilon.$$

Thus, $T - P$ names are, within small $\bar{d}$ error, copies of β and a small proportion of spacer symbols. ∎

It turns out that with a careful choice of P and the stack sequence a more rigid structure for $T - P$ names is ensured. This well-known construction is described in King [Ki]; we will simply sketch here what is possible. We will make essential use of these properties in what follows.

First, one may simultaneously construct a new sequence of *refining* stacks

$$\tilde{R}_1 < \tilde{R}_2 < \cdots < \tilde{R}_n < \cdots \qquad (\mu(\tilde{E}_n) \to 0)$$

which generate the σ-algebra, and a two set generating partition $P = \{P_0, P_1\}$ so that each stack level is contained in an atom of P (*i.e.* $P < \tilde{R}_n$). The implications of this for $T - P$ names follows.

(2.3) There exists a sequence of strings of zeros and ones (n-blocks) so that for all n, an $(n+1)$-block is a concatenation of n-blocks interspersed with spacers (for convenience always the symbol "1") and, $\epsilon > 0$ given, for almost every $x \in \{0,1\}^{\mathbf{Z}}$, for all large L, x_0^L is a concatenation of n-blocks interspersed with spacers occupying at most a proportion ϵ of the indices in x_0^L.

In case T is aperiodic we may also ensure that the base $\tilde{A}_1^{(n)}$ of the stack $\tilde{R}_n = \{\tilde{A}_1^{(n)}, \tilde{A}_2^{(n)}, \ldots, \tilde{A}_{b_n}^{(n)}, \tilde{E}_n\}$ is measurable with respect to $\bigvee_0^{b_n-1} T^{-i}P$; the implication for n-blocks being:

(2.4) The position of n-blocks inside the $(n+1)$-blocks is uniquely determined.

(2.5) For almost every $x \in \{0,1\}^{\mathbf{Z}}$, since $\mu(\tilde{E}_n) \downarrow 0$ we also see that for all large enough n, the time–zero coordinate of x, x_0 is interior to the appearance of an n-block in $x_{-\infty}^{\infty}$. We shall denote this "time-zero" n-block by $B_n(x)$.

One way to obtain 2.4) is to ensure that the $T - P$ name of $x \in \tilde{A}_1^{(n)}$ begins with a unique string:

(2.6) We may assume an n-block begins with "1" followed by a string of n "0"'s followed by a "1", and that an n-block ends with a "1". All spacers are assigned the label "1".

In practice, a rank 1 example is often constructed by giving a recursive definition of the n-blocks using a fixed symbol for spacers and obtaining the shift invariant measure μ on cylinders A as the limiting frequency of the appearance of A in an n-block. This point of view is nicely described by Kalikow in [Ka]. One thereby obtains 2.3) and 2.5) but not necessarily the conditions 2.4) and 2.6) for the time–zero partition.

We conclude this section with the observation that, in shift setting for aperiodic rank 1 T, without loss of generality, most appearances of an n-block in $x_{-\infty}^{\infty}$ will be followed by a spacer string whose length is a "small" fraction of the length of an n-block. To this end we define, if x_0 is in a time-zero n-block,

$$l_n(x) = \# \text{ of spacer symbols between } B_n(x) \text{ and the next } n\text{-block to the right of } B_n(x) \text{ in } x_{-\infty}^{\infty}.$$

Set $l_n(x) = 0$ if x_0 is not in an n-block. Let $\delta_0 > 0$ be fixed and set

$$H_n = \{x \in \tilde{A}_1^{(n)} | \ l_n(x) > \delta_0 |B_n(x)|\}.$$

Since, for $x \in \tilde{A}_1^{(n)}$

$$\mu(\tilde{E}_n) \geq \delta_0 |B_n(x)| \mu(H_n) = \delta_0 |B_n(X)| \mu(\tilde{A}_1^{(n)}) \frac{\mu(H_n)}{\mu(\tilde{A}_1^{(n)})}$$

we obtain

$$\frac{\mu(\tilde{E}_n)}{\delta_0 |B_n(x)| \mu(\tilde{A}_1^{(n)})} \geq \frac{\mu(H_n)}{\mu(\tilde{A}_1^{(n)})} \geq 0.$$

Since $|B_n(x)| \mu(\tilde{A}_1^{(n)}) = 1 - \mu(\tilde{E}_n) \to 1$, the left-hand side converges to zero. For large enough n, only a small fraction of x in $\tilde{A}_1^{(n)}$, and hence in $\tilde{R}_n$, will have their time-zero n-block followed by a spacer string longer than $\delta_0 |B_n(x)|$. We may now apply the Borel–Cantelli Lemma to obtain

LEMMA 2.7: *Let $\delta_0 > 0$ be given. Then we may choose a subsequence of the n-blocks B_{n_k} so that, for almost every $x \in \{0,1\}^{\mathbf{Z}}$, for all sufficiently large k, the time-zero n-block for x, $B_{n_k}(x)$ is followed by a spacer string of length less than $\delta_0 |B_{n_k}(x)|$.*

3

We describe a general construction which embeds an abstract dynamical system as a factor of the natural extension of a piecewise monotone and continuous interval map. The main idea here appears to be new and it is hoped that it will find application beyond its use in this article.

The following condition will be imposed on our abstract dynamical system.

Definition 3.1: Let $\delta > 0$. We say that the process $(X, \mathcal{B}, \mu, T, P)$ satisfies the $\bar{d} > \delta$ **property** if there is an $X_0 \subseteq X$, $\mu(X_0) = 1$, so that if $x, y \in X_0$, $x \neq y$ then either

$$\limsup_{n \to \infty} \bar{d}(x_0^n, y_0^n) > \delta$$

or

$$\limsup_{n\to\infty} \bar{d}(x_{-n}^{0}, y_{-n}^{0}) > \delta$$

where $x_{-\infty}^{\infty}$ and $y_{-\infty}^{\infty}$ are the $T-P$ names of x and y respectively. ∎

Evidently, the above is a condition on a process. It is not hard to show that this is in fact a property of the automorphism only, but since we shall not be needing this we will say that the dynamical system $(X, \mathcal{B}, \mu, T)$ satisfies the $\bar{d} > \delta$ property if there exists a generating partition P so that the process $(X, \mathcal{B}, \mu, T, P)$ satisfies $\bar{d} > \delta$. The main tool we shall use in the constructions to follow is

THEOREM 3.2: *Let $(Y, \mathcal{F}, \nu, S)$ be a dynamical system satisfying the $\bar{d} > \delta$ property with respect to a generating partition containing l atoms. Then there exists a p.m.c. (with respect to l subintervals) map $g: [0,1] \to [0,1]$ which satisfies 1.1), 1.2), 1.3) and 1.6), which is Lebesgue–measure–preserving and whose natural extension is measurably isomorphic to a direct product of the transformation S and a Bernoulli shift.*

Let $R = \{R_0, R_1, \ldots, R_{l-1}\}$ be the generating partition for S. Let $(\Omega, \mathcal{G}, p, \sigma)$ be a Bernoulli shift with independent generator $Q = \{Q_0, Q_1, \ldots, Q_{l-1}\}$ satisfying

$$p(Q_0) = 1 - \frac{\delta}{3}, \quad p(Q_1) = p(Q_2) = p(Q_{l-1}) = \frac{\delta}{3(l-1)}.$$

Form the product dynamical system

$$(X, \mathcal{B}, \mu, T) = (Y \times \Omega, \mathcal{F} \times \mathcal{G}, \nu \times p, S \times \sigma)$$

and define a measurable partition of X as follows: $P = \{P_0, P_1, \ldots, P_{l-1}\}$ with

$$P_j = \bigcup_{s+t=j(\mathrm{mod}\ l)} R_s \times Q_t, \qquad j = 0, 1, \ldots, l-1.$$

There is a simple formula to construct the $T-P$ name of $x = (y, \omega)$ given the $S-R$ name of y and the σ–Q name of ω:

$$x_i = y_i + \omega_i (\mathrm{mod}\ l); \quad i \in \mathbb{Z}.$$

We will need the following:

LEMMA 3.3: *The partition P is a generator for T.*

Proof: Let $Y_0 \subseteq Y$ be the set of full measure given by the $\bar{d} > \delta$ property and let $\Omega_0 \subseteq \Omega$ be the set of full measure whose points are separated by $\bigvee_{-\infty}^{\infty} \sigma^{-i}Q$ and are generic for Q. we will show that the $T - P$ names of $x = (y, \omega)$ and $x' = (y', \omega')$ with $x \neq x'$ in $Y_0 \times \Omega_0$ are distinct.

Suppose not. Observe first that $y \neq y'$ since $\{\sigma^{-i}Q\}_{i\in\mathbb{Z}}$ separates the points of Ω_0 and for each s there is exactly one t so that $s + t = j$.

Next, obtain a sequence of integers $n_k \to \infty$ (or $-n_k \to -\infty$, the argument will be the same) so that $\bar{d}(y_0^{n_k}, {y'}_0^{n_k}) > \delta$ for all k. Conclude that $\bar{d}(\omega_0^{n_k}, {\omega'}_0^{n_k}) > \delta$ since under our supposition, ω_i and ${\omega'}_i$ will differ on every index i such that $y_i \neq {y'}_i$.

But that gives a contradiction, since for sufficiently large k, the number of indices $i \in [0, n_k)$ for which ω_i and ${\omega'}_i$ will both be zero exceeds $\left[1 - \left[\frac{2\delta}{3} + \frac{\delta}{3}\right]\right] n_k = (1-\delta) n_k$ (the extra $\frac{\delta}{3}$ to take care of the difference between frequency of zero and $\mu(Q_0)$) whenever ω and ω' are generic for Q. ∎

Now, for $j = 0, 1, \ldots, l-1$ and $x \in X$ define

$$\varphi_j(x) = E\left[\chi_{P_j} \mid \bigvee_{i=1}^{\infty} T^{-i}P\right](x) = \lim_N E\left[\chi_{P_j} \bigvee_{i=1}^{N} T^{-i}P\right](x) \overset{\text{a.e.}}{=} \lim_N \varphi_j^{(N)}(x).$$

LEMMA 3.4: *For all j and a.e. $x \in X$, $\varphi_j(x) \in \left[\frac{\delta}{3(l-1)}, 1 - \frac{\delta}{3}\right]$.*

Proof: It is enough to show this bound for $\varphi_j^{(N)}(x)$. Let A be an atom of $\bigvee_{i=1}^{N} T^{-i}P$. Observe that for each atom $a_k \in \bigvee_{i=1}^{N} S^{-i}R$ there is exactly one atom $\beta_k \in \bigvee_{i=1}^{N} \sigma^{-i}Q$ such that $\mu((\alpha_k \times \beta_k) \cap A) > 0$. Moreover, using this we may write A as a disjoint union

$$A = \bigcup_{k=1}^{l^N} (\alpha_k \times \beta_k).$$

Expanding,

$$\begin{aligned} E\left[\chi_{P_j} \mid \bigvee_{i=1}^{N} T^{-i}P\right](x) &= \sum_A \chi_A(x)\mu(P_j|A) \\ &= \sum_A \chi_A(x)\left[\sum_k \mu(\alpha_k \times \beta_k|A)\mu(P_j|\alpha_k \times \beta_k)\right]; \end{aligned}$$

we see that it is enough to obtain the advertised bound for each term $\mu(P_j|\alpha_k \times \beta_k)$. But

$$\begin{aligned}\mu(P_j|\alpha_k \times \beta_k) &= \sum_i \mu\Big[P_j \cap (R_i \times \Omega)|\alpha_k \times \beta_k\Big] \\ &= \sum_i \mu\Big[R_i \times Q_{s(i)}|\alpha_k \times \beta_k\Big]\end{aligned}$$

where $s(i)$ is that $s \in \{0,1,\ldots,l-1\}$ satisfying $s+i=j(\text{mod } l)$.

Finally using the independence we may rewrite the above sum as

$$\sum_i \frac{\nu(R_i\cap\alpha_k)p(Q_{s(i)}\cap\beta_k)}{\nu(\alpha_k)p(\beta_k)} = \sum_i \frac{\nu(R_i\cap\alpha_k)}{\nu(\alpha_k)}p(Q_{s(i)}) \in \Big[\frac{\delta}{3(l-1)}, 1-\frac{\delta}{3}\Big]. \qquad \blacksquare$$

In applying these Lemmas to a proof of Theorem 3.1, we assume the reader is familiar with the generalized baker's transformation (g.b.t.) representation of an abstract measure-preserving automorphism as a Lebesgue-measure-preserving automorphism of the unit square $\mathcal{S} = [0,1] \times [0,1]$. Those wishing for a more complete discussion are referred to [Bo1] where this construction is described and all of the properties we shall need are discussed in detail.

We summarize the results of applying this representation to our automorphism T with respect to the partition P:

(3.5) Let $\varphi_j : [0,1] \to [0,1]$, $j = 0,1,\ldots,l-1$ be a collection of measurable functions satisfying

$$\sum \varphi_j = 1.$$

Let $\underline{f} = \langle\varphi_0,\varphi_1,\ldots,\varphi_{l-1}\rangle$ and define $P_{\underline{f}} = \{P_{\underline{f}}^{(0)}, P_{\underline{f}}^{(1)},\ldots,P_{\underline{f}}^{(l-1)}\}$, the "natural" partition of $\mathcal{S}$ induced by $\underline{f}$, where

$$P_{\underline{f}}^{(j)} = \Big\{(x,y)\in\mathcal{S}| \quad \sum_{i<j}\varphi_i(x) \le \sum_{i\le j}\varphi_i(x)\Big\}.$$

The φ_i, in the terminology of [Bo1] are the "cutting functions". The action of $T_{\underline{f}}$ is best described in terms of $T_{\underline{f}}^{-1}$, which maps an atom $P_{\underline{f}}^{(j)}$ onto the vertical column $C_j = \big[\sum_{i<j}\int_0^1\varphi_i(x)dx, \quad \sum_{i\le j}\int_0^1\varphi_i(x)dx\big] \times [0,1]$ in a Lebesgue–measure–preserving way and so as to send vertical fibers inside $P_{\underline{f}}^{(j)}$ onto full vertical fibers in C_j. It is easy to write an explicit formula for such a $T_{\underline{f}}$ but we shall not need this. The functions φ_j may be chosen in such a way that the process $(T_{\underline{f}}, P_{\underline{f}})$ is isomorphic to the process (T,P).

(3.6) The σ-algebra $(P_{\underline{f}})_{-\infty}^{-1} = \bigvee_{-\infty}^{-1} T_{\underline{f}}^i P_{\underline{f}}$ is measurable with respect to $\mathcal{V}$, the σ-algebra of vertical fibres on $\mathcal{S}$. Let $\pi_1 : \mathcal{S} \to [0,1]$ be first coordinate projection. The cutting functions can be shown to be measurable with respect to $\pi_1(P_{\underline{f}})_{-\infty}^{-1}$ and hence

$$\varphi_j(x) = \mathrm{E}\left[\chi_{P_{\underline{f}}^{(j)}} \,|(P_{\underline{f}})_{-\infty}^{-1}\right](\pi_1^{-1}x).$$

(3.7) In view of Lemma 3.4 we see that the φ_j satisfy

$$\frac{\delta}{3(l-1)} \leq \varphi_j \leq 1 - \frac{\delta}{3}.$$

In particular, $P_{\underline{f}}$ is a generator for $T_{\underline{f}}$ and the isomorphism of processes in 3.5) is in fact measurable isomorphism of the two dynamical systems (X, T, μ) and $(\mathcal{S}, T_{\underline{f}}, \lambda)$.

(3.8) We may specify that, restricted to each column C_j, $T_{\underline{f}}$ preserves order in each coordinate. Let I_j be the base of the column C_j, *i.e.* $I_j \times [0,1] = C_j$. Let $g_j : I_j \to [0,1]$ be $g_j = \pi_1(T_{\underline{f}}(\pi^{-1}x))$ and define $g : [0,1] \to [0,1]$ by $g|_{I_j} = g_j$. g is the Lebesgue–measure–preserving factor of $T_{\underline{f}}$ on vertical fibers of $\mathcal{S}$ and is our advertised interval map. We collect the following facts about g.

(3.9) The intervals I_j satisfy 1.1), $j = 0, 1, \ldots, l-1$. The lengths of the I_j satisfy

$$\frac{\delta}{3(l-1)} \leq |I_j| \leq 1 - \frac{\delta}{3}.$$

(3.10) In view of 3.5) and 3.8) above g satisfies 1.2), indeed each g_j is continuous and monotone increasing on I_j.

(3.11) For each j, $g(I_j) = [0,1]$, in particular g is a Markov Map (property 1.7)) of the unit interval.

(3.12) For each j,

$$g_j'(x) = \frac{1}{\varphi_j(g(x))} \geq \frac{1}{1 - \frac{\delta}{3}} > 1 \quad \text{for almost every } x \in I_j.$$

Thus g is expanding (property 1.3)).

(3.13) Since g is Lebesgue–measure–preserving we have

$$\sum_{y \in g^{-1}(x)} \frac{1}{g'(y)} = 1,$$

and if g is ergodic, Lebesgue measure is the unique (amongst all measures absolutely continuous with respect to Lebesgue measure) invariant measure for g.

(3.14) Since $P_{\underline{f}}$ is a generator for $T_{\underline{f}}$, $(T_{\underline{f}}, \lambda^2)$ is the natural extension of the system $g, \lambda)$. (In general, the factor of $T_{\underline{f}}$ on $\bigvee_{-\infty}^{\infty} T_{\underline{f}}^i P_{\underline{f}}$ gives the natural extension for g.)

4

We obtain our examples by specifying the process (S, Q) in the construction from Section 3. In this section we will discuss two preliminary cases before proceeding, in Section 5, with the deeper mixing examples.

Example 4.1: An interval map satisfying 1.1) – 1.3), and 1.6) which is ergodic but not weak-mixing with respect to its unique absolutely continuous invariant measure. ∎

Let $Y = \{0, 1\}$, $\mathcal{B}$ = discrete σ-algebra on Y and ν = equidistributed probability measure on the two points. Let S be the two point flip: $S(0) = 1$, $S(1) = 0$. Let $R = \{\{0\}, \{1\}\}$. Evidently (S, R) satisfies the $\bar{d} > \delta$ property for every $\delta < 1$. Fix $\delta = \frac{3}{4}$.

The product automorphism T is ergodic but not weak-mixing, possessing a rotation factor. Conclude that the interval map g from Section 3 is ergodic, being a factor of T, but cannot be weak-mixing, since its natural extension, T, is not.

Example 4.2: An interval map satisfying 1.1) – 1.3), and 1.6) which is weak-mixing but not mixing with respect to its unique absolutely continuous invariant measure. ∎

The automorphism S is chosen to be Chacón's automorphism, first defined and studied in [C1] using a cutting-and-stacking description. It will be more convenient for us to use the rank 1 shift description of S. The symbol set is $\{0, 1\}$. Recall from Section 2 that it suffices to specify the m-blocks:

$$B_0 = 0, \qquad B_{n+1} = B_n \otimes B_n \otimes 1 \otimes B_n.$$

It is a routine check that the rank 1 conditions 2.3), 2.4) and 2.5) are satisfied. Let R be the time zero partition of $Y \subseteq \{0,1\}^{\mathbf{Z}}$ and recall the convention $b_n = |B_n|$.

We say that B_n **appears at** i_0 in the $S-R$ name of $x \in Y$ if $x_{i_0}^{i_0+b_n-1} = B_n$. Given two n-blocks $B_n(x)$ and $B_n(y)$ appearing at $i_0(x)$ in $x_{-\infty}^{\infty}$ and $i_0(y)$ in $y_{-\infty}^{\infty}$ respectively we say these two blocks **line up** if $i_0(x) = i_0(y)$ and we say they **overlap** if

$$i_0(y) - b_n + 1 \leq i_0(x) \leq i_0(y) + b_n - 1.$$

Finally, we say that these two n-blocks **overlap substantially** (for the purposes of this example only) if they overlap by more than $b_n/3$ indices, that is,

$$i_0(y) - b_n + 1 + \frac{b_n}{3} < i_0(x) < i_0(y) + b_n - 1 - \frac{b_n}{3}.$$

We begin by showing that two n-blocks which do not line up, but overlap substantially must generate an **a priori** lowerbounded amount of $\bar{d}$ disagreement over this overlap. We will be arguing in a "coordinate free" manner where possible, using two copies of B_n, $\overline{B_n}$ and $\underline{B_n}$, with $\overline{B_n}$ lying above $\underline{B_n}$. We enumerate the three copies of B_{n-1} inside B_n, starting from the left as $B_{n-1}^{(1)}, B_{n-1}^{(2)}$ and $B_{n-1}^{(3)}$. When we wish to describe which copy of B_n these subblocks occupy we shall write $\overline{B_{n-1}^{(i)}}$ or $\underline{B_{n-1}^{(i)}}$ as appropriate. Finally, given the two copies $\overline{B_n}$ and $\underline{B_n}$ we will agree that $\bar{d}(B_n, S^j B_n)$ refers to the distance measured over the overlap of $\overline{B_n}$ and $\underline{B_n}$ shifted j indices to the right relative to $\overline{B_n}$.

LEMMA 4.3: *Let $B = B_n$ $(n \geq 1)$ and let $1 \leq j \leq \frac{2}{3} b_n$. Then*

$$\bar{d}(B, S^j B) > \tfrac{1}{7}.$$

This will certainly be true if we can prove the stronger

LEMMA 4.4: *Let $B = B_n$ $(n \geq 1)$. Then*

(a) *$\bar{d}(B, S^j B) > \frac{1}{7}$ if $1 \leq j \leq \frac{2}{3}|B|$.*

(b) *If $\bar{d}(B, S^j B) \leq \frac{1}{7}$ then $\bar{d}(B, S^{j\pm 1} B) > \frac{1}{2}$ where we allow $j = 0, 1, \ldots, b_{n-1}$.*

(c) *If $j = b_k$ or $j = b_k + 1$ for some $k < n$ then $\bar{d}(B, S^j B) > \frac{1}{4}$.*

Proof: These three statements are easily verified for the case $n = 1$ by direct calculation. We assume $n \geq 2$ and the result true for $1 \leq k < n$. Note that $b_n \geq 13$ and $b_{n-1} \geq 4$.

CASE 1: j=1. There are 3 substantial overlaps of $(n-1)$ blocks each yielding $\bar{d} > \frac{1}{2}$ by application of b). Of the 3 remaining indices, 2 record a disagreement (the two spacer "1"'s) yielding error rate $\frac{2}{3}$ over this collection of indices. Combining we have $\bar{d} > \frac{1}{2}$.

CASE 2: $1 < j < \frac{2}{3}b_{n-1}$. There are 3 substantial $(n-1)$-block overlaps each of which gives $\bar{d} > \frac{1}{7}$ by a). There are two (possibly) insubstantial $(n-1)$-block overlaps and the two indices corresponding to spacer "1"'s remaining. If each $(n-1)$-block (possibly) insubstantial overlap yields $\bar{d} > \frac{1}{7}$ then we proceed as follows:

If one of the spacer "1"'s aligns with a zero we easily calculate $\bar{d} > \frac{1}{7}$ on the entire overlap. If $\bar{d} = 0$ measured over the two spacer indices we proceed as follows. Suppose k is the length of (all 3) substantial $(n-1)$-block overlaps, l and $(l-1)$ respectively, the length of the (possibly) insubstantial ones. The number of indices of disagreement on, say, an interval of length l is *strictly* greater than $\frac{1}{7}l$ and so is (an integer) $\geq \frac{1}{7}(l+1)$. Counting errors this way we obtain $\geq \frac{1}{7}(3k+3+2l+1)$ over the entire overlap of length $3k+3+2l+1$. Conclude, again $\bar{d} > \frac{1}{7}$ on the n-block overlap. Let us agree to call the above argument **absorbing $\bar{d} = 0$ over the spacers** and observe that we need one more good overlap interval than spacer index to make it work.

Next, if one of the (possibly) insubstantial overlaps yields $\bar{d} \leq \frac{1}{7}$, since the other overlap appears shifted by one, it sees $\bar{d} > \frac{1}{2}$. This yields at least $\frac{1}{2}l$ disagreements over the two overlaps of total length $2l-1$ (lengths l and $(l-1)$, respectively) giving $\bar{d} \geq \frac{l}{4l-2} > \frac{1}{4}$. Again, we have enough intervals to absorb $\bar{d} = 0$ for the two spacer "1"'s.

CASE 3: $\frac{2}{3}b_{n-1} \leq j < b_{n-1}$. The $\overline{B_{n-1}^{(1)}}/\underline{B_{n-1}^{(1)}}$ overlap is not substantial. Choose $0 \leq k < n-1$ to be smallest so that the last B_k block in $\overline{B_{n-1}^{(1)}}$ covers the above mentioned overlap. There are now two possibilities. If the last B_k in $\overline{B_{n-1}^{(1)}}$ and the first B_k in $\underline{B_{n-1}^{(1)}}$ do not line up then they overlap substantially and we obtain $\bar{d} > \frac{1}{7}$ on all three insubstantial overlaps. Absorb $\bar{d} = 0$ over the spacers if necessary. Otherwise these two copies of B_k line up and the substantial B_{n-1} overlaps are shifts by b_k and b_k+1 respectively, $k < n-1$. Applying c) we obtain $\bar{d} > \frac{1}{4}$ on these two substantial overlaps. The spacer "1" in $\overline{B_n^{(1)}}$ aligns with "0" in $\underline{B_n^{(1)}}$. The substantial overlaps plus spacer indices amount to at least $\frac{4}{7}$ of the total overlap length. Conclude, again, $\bar{d} > \frac{1}{7}$.

CASE 4: $j = b_{n-1}$ or $j = b_{n-1}+1$. One easily obtains $\bar{d} > \frac{1}{4}$ as one of the B_{n-1} blocks lines up and the other is shifted by one. In both cases any spacer "1" is aligned with "0".

CASE 5: $b_{n-1}+1 < j < b_{n-1} + \frac{2}{3}b_{n-1}$. Two of the $(n-1)$-block overlaps are substantial yielding $\bar{d} > \frac{1}{7}$ there. Again choose k smallest so the first B_k in $\overline{B^{(3)}_{n-1}}$ covers the $\overline{B^{(3)}_{n-1}}/\underline{B^{(1)}_{n-1}}$ overlap. If this B_k block does not align with the last B_k block in $\underline{B^{(1)}_{n-1}}$ then the overlap is substantial (for k) and we obtain $\bar{d} > \frac{1}{7}$. One absorbs $\bar{d} = 0$ over the single spacer with the three overlap intervals if necessary. Otherwise these B_k blocks line up and the two $(n-1)$-block shifts are b_k+1 and b_k respectively. The spacer "1" aligns with "1", nevertheless we calculate, using c) from the hypothesis, with l the length of the $\overline{B^{(2)}_{n-1}}/\underline{B^{(1)}_{n-1}}$ overlap

$$\bar{d} \geq \frac{\frac{1}{4}(2l+3)}{2l+2+b_k} \geq \frac{\frac{1}{4}(2l+3)}{2l+2+\frac{1}{2}l}$$

since $l \geq 2b_k$. The latter is clearly $> \frac{1}{7}$.

CASE 6: $b_{n-1} + \frac{2}{3}b_{n-1} \leq j < 2b_{n-1}$. The $\overline{B^{(3)}_{n-1}}/\underline{B^{(1)}_{n-1}}$ overlap is substantial yielding $\bar{d} > \frac{1}{7}$ there. If both remaining two overlaps yield $\bar{d} > \frac{1}{7}$ one may absorb $\bar{d} = 0$ over the spacer. Otherwise, if one insubstantial overlap yields $\bar{d} \leq \frac{1}{7}$, the other yields $\bar{d} > \frac{1}{2}$ being a shift by one index (apply b)). Counting disagreements we get at least $\frac{1}{2}l$ (l the length of the longer overlap) over the $2l$ indices plus the spacer. One therefore gets $\bar{d} > \frac{1}{4}$ there, and hence $\bar{d} > \frac{1}{7}$ over the whole n-block overlap.

CASE 7: $j = 2b_{n-1}$. Obtain $\bar{d} > \frac{1}{2}$ easily.

This finishes the proof of a) for $B = B_n$. If $\bar{d}(B, S^jB) \leq \frac{1}{7}$ then either $j = 0$ or $j \geq 2b_{n-1}+1$. The first value is handled by case 1 and for the second value, note the overlap is entirely contained in the $\overline{B^{(3)}_{n-1}}/\underline{B^{(1)}_{n-1}}$ overlap and the induction hypothesis applies directly. b) has been shown. To obtain c) observe that if $k = n-1$ this was handled by Case 4. Otherwise the overlap consists of three $(n-1)$-blocks shifted by b_k ($k < n-1$) yielding $\bar{d} > \frac{1}{4}$, one aligned B_k block ($\bar{d} = 0$) and one B_k block shifted against a B_k block by one unit. One of the spacer "1"'s aligns with "0". Evidently $\bar{d} > \frac{1}{4}$. This verifies c) and completes the proof. ∎

Let $x \in Y \subseteq \{0,1\}^{\mathbb{Z}}$ and let $m, n \in \mathbb{Z}$. We say that an n-block in $x_{-\infty}^{\infty}$ covers x_n^m if there exists $n_0 \leq n \leq m \leq m_0$ so that $x_{n_0}^{m_0} = B_n$. It remains to lift the $\bar{d}$ separation of n-blocks to full names.

LEMMA 4.5: *The process (S, R) satisfies $\bar{d} > \frac{1}{42}$.*

Proof: Let $x \neq y \in Y$ be fixed. We may as well assume $x_0 \neq y_0$ for we can always shift the picture: If I_n is a sequence of intervals of integers starting at zero so $\lim_{n\to\infty}|I_n| = +\infty$ and

$$\limsup_n \bar{d}\left[S^{n_0}x|_{I_n} \quad S^{n_0}y|_{I_n}\right] > \delta$$

then

$$\limsup_n \bar{d}\left[x|_{I_n}, y|_{I_n}\right] > \delta.$$

Obtain, for all sufficiently large n, time-zero n-blocks $B_n(x)$ and $B_n(y)$ covering x_0 and y_0 respectively. Evidently these n-blocks do not line up.

Case 1. There are ∞-many n so that $B_n(x)$ and $B_n(y)$ are shifted by one index with respect to each other. Enumerate such n by $\{n_k\}_{k=1}^{\infty}$. By dropping to a further subsequence we obtain intervals of integers

$$I_{n'_k} = [\alpha_{n'_k}, \beta_{n'_k}] \cap \mathbb{Z}$$

satisfying either

(i) for all k, $I_{n'_k} \subseteq \mathbb{Z}^+ \cup \{0\}$ and forms the indices of overlap between $B^{(3)}_{n'_k-1}(x)$ and $B^{(3)}_{n'_k-1}(y)$

(ii) for all k, $I_{n'_k} \subseteq \mathbb{Z}^- \cup \{0\}$ and forms the indices of the $B^{(1)}_{n'_k-1}(x)$ and $B^{(1)}_{n'_k-1}(y)$ overlap.

In both cases $|I_{n'_k}| = b_{n'_k-1} \to +\infty$. Also in both cases

$$\max\{|n| \ |n \in I_{n'_k}\} \leq b_{n'_k} - 1 = 3b_{n_k-1}.$$

In the first instance set $N_k = \beta_{n'_k}$ and observe $\bar{d}(x_0^{N_k}, y_0^{N_k}) > \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. In the second instance, set $N_k = \alpha_{n'_k}$ obtaining

$$\bar{d}(x_{N_k}^0, y_{N_k}^0) > \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

from Lemma 4.4 b).

Case 2. For all sufficiently large n, $B_n(x)$ and $B_n(y)$ appear shifted by more than one index relative to each other. Let $B_n^+(x)$ and $B_n^+(y)$ be the first n-blocks appearing to the right of the previously defined $B_n(x)$ and $B_n(y)$. Evidently $B_n^+(x)$ and $B_n^+(y)$ do not line up. We define intervals of indices $I_n = [\alpha_n, \beta_n] \cap \mathbb{Z}^+$ as follows: If $B_n^+(x)$ and $B_n^+(y)$ overlap substantially then I_n is defined by this overlap. If not, then I_n is defined by the overlap of either $B_n(x)$ and $B_n^+(y)$ or of $B_n^+(x)$ and $B_n(y)$, one of which must be a substantial n-block overlap. We thereby obtain intervals I_n satisfying

(i) $I_n \subseteq \mathbb{Z}^+$.
(ii) For each n, I_n is the set of indices of a substantial n-block overlap.
(iii) $|I_n| > \frac{1}{3}b_n$ and $\max\{k \in I_n\} \leq 2b_n$.

Setting $N_n = \beta_n$ conclude $N_n \to \infty$ and

$$\bar{d}(x_0^{N_n}, y_0^{N_n}) > \frac{1}{6} \cdot \frac{1}{7} = \frac{1}{42}$$

where we have used Lemma 4.3 to obtain $\bar{d} > \frac{1}{7}$ on the interval I_n. ∎

Remark 4.5:

(i) The $\bar{d} > \delta$ property for Chacón's Automorphism has not been observed before in the literature.
(ii) It is well known (and easy to prove, see [C1]) that S is weak-mixing but not mixing. The same will be true for T, the product of S and a Bernoulli shift.

∎

The interval map g constructed in Section 3 will have Lebesgue measure as its unique (abs. cont.) invariant measure, with respect to which g is weak-mixing but not mixing, since the latter property lifts to natural extensions.

5

In this section we discuss our final example in which S is a mixing rank 1 automorphism. Ornstein [Or] was the first to show that such transformations exist using a "random spacer" construction. With some effort, using the ideas of King and Weiss [Ki We], one may modify the Ornstein construction obtaining a mixing rank 1 with $\bar{d} > \frac{1}{4}$. It turns out, however, that one need not to be so industrious.

THEOREM 5.1: *Let S be a rank 1 transformation which is mixing. Then S satisfies $\bar{d} > \delta$.*

Remark 5.2: (i) Having chosen S as above the results of Section 3 are invoked to produce an automorphism T (product of S and a Bernoulli shift) which is mixing but not a K-automorphism, possessing a non-trivial zero-entropy factor. It is in fact only the zero-entropy property of S we need — any mixing, zero-entropy automorphism with $\bar{d} > \delta$ would serve for S.

(ii) The interval map g associated to T in Section 3 has Lebesgue measure as its unique (amongst those abs. cont. to Lebesgue measure) invariant measure, and is mixing but not exact (in the terminology of endomorphisms) since its natural extension is not a K-automorphism.

In what follows S will be a rank 1 mixing automorphism viewed as a μ-invariant shift on the space X of sequences of symbols from A with time-zero partition R. As discussed in Section 2 there is no loss of generality in assuming $A = \{0, 1\}$, and the rank 1 properties 2.3) through 2.6) there. ∎

Once again, we first obtain a lower bound on the $\bar{d}$ separation of shifted n-blocks. The central idea is contained in the following

PROPOSITION 5.3: *Let $\epsilon_0 > 0$ be fixed. Then there exist $0 < M < \infty$ and $\delta > 0$ so that for all large N, if α and β are two substrings of an N-block B_N beginning at i and $i + m$ respectively, with $m \geq M$ and $|\alpha| = |\beta| \geq \epsilon_0 b_N$, then*

$$\bar{d}(\alpha, \beta) > \delta.$$

Proof: By contradiction. To be precise, assume we may find an $\epsilon_0 > 0$ and a sequence of blocks B_{N_k} and substrings α_k and β_k of B_{N_k} satisfying:

5.4) $b_{N_k} = |B_{N_k}| \to \infty$.

5.5) $|\alpha_k| = |\beta_k| = l_k \geq \epsilon_0 b_{N_k}$.

5.6) a_k appears at $i_k \in [0, b_{N_k} - 1]$ while β_k appears at $i_k + m_k$ with $m_k \overset{k\to\infty}{\rightarrow} +\infty$.

5.7) $\bar{d}(\alpha_k, \beta_k) = \delta_k \overset{k\to\infty}{\rightarrow} 0$.

We will show that these conditions contradict the mixing of S.

Fix a positive integer n (to be specified later) and let $C_1, \ldots, C_p$ be a listing of the atoms of $\bigvee_{i=0}^{n-1} S^{-i}R$. We identify, in the usual manner, atoms with strings (of zeros and ones) of length n, which we shall also denote by C_j. Choose an atom, say $C_{j(k)}$, which appears in α_k at least $\mu(C_{j(k)})(l_k - n)$ times. This is possible since no string corresponding to a null set in $\bigvee_{i=0}^{n-1} S^{-i}R$ can appear in α_k. Fix $x \in X$ generic. We will estimate $\mu(S^{-m_k}C_{j(k)} \cap C_{j(k)})$ from below by examining the frequency of appearance of this cylinder in the $S-R$ name of x over the interval of indices $[0, N] \cap \mathbb{Z}$, more precisely by counting those appearances of $C_{j(k)}$ which occur in α_k, sitting at i_k inside blocks B_{N_k} with $C_{j(k)}$ also at $i_k + m_k$, all in the name of x. Let $K = K(N, x)$ be the number of full N_k blocks appearing in x_0^N, obtaining:

$$\begin{aligned}
\mu(S^{-m_k}C_{j(k)} \cap C_{j(k)}) &\geq \frac{K\{\mu(C_{j(k)})(l_k - n) - n\bar{d}(\alpha_k, \beta_k)l_k\}}{N} - \epsilon_1 \\
&= \frac{\mu(C_{j(k)})(l_k - n) - n\delta_k l_k}{b_{N_k}}\left[\frac{K b_{N_k}}{N}\right] - \epsilon_1 \qquad (5.8)\\
&\geq \left\{\mu(C_{j(k)})\left[\frac{l_k - n}{b_{N_k}}\right] - \frac{n\delta_k l_k}{b_{N_k}}\right\}[1 - \mu(B_{N_k}^c) - \epsilon_2] - \epsilon_1,
\end{aligned}$$

where ϵ_1 and ϵ_2 allow for exchange of the measure of a cylinder set with its frequency of appearance in x_0^N and may be arranged arbitrarily small for sufficiently large N.

Since there are only finitely many values possible for the integer $j(k)$, without loss of generality we may assume (having dropped to a subsequence of the N_k) that all k, $j(k) = j_0 \in \{1, 2, \ldots, p\}$. For fixed n, for all large enough k we have:

$$\frac{l_k - n}{b_{N_k}} > \frac{\epsilon_0}{2} \qquad \text{by 5.5)} \tag{5.9}$$

$$\frac{n\delta_k l_k}{b_{N_k}} \leq \epsilon_4 \qquad \text{by 5.7)} \tag{5.10}$$

$$[1 - \mu(B_{N_k}^c) - \epsilon_2] \geq 1 - \epsilon_5. \tag{5.11}$$

Conclude, for fixed n, for all large enough k, from (5.8) - (5.11)

$$\mu(S^{-m_k}C_{j_0} \cap C_{j_0}) \geq [\frac{\epsilon_0}{2}\mu(C_{j_0}) - \epsilon_4](1 - \epsilon_5) - \epsilon_1 \geq \frac{\epsilon_0}{3}\mu(C_{j_0})$$

for judiciously chosen parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_5$. Since $m_k \to \infty$ and S is mixing we see from the above inequality that, necessarily, $\mu(C_{j_0}) \geq \frac{\epsilon_0}{3}$. But, n was arbitrary and

$$M_n = \max\left\{\mu(C) | \quad C \in \bigvee_{i=0}^{n-1} S^{-i}R\right\}$$

satisfies $\lim_{n\to\infty} M_n = 0$ in order for (S,μ) to be non-atomic. Hence the contradiction. ∎

It remains to handle substrings α and β shifted by fewer than M indices. In the following, we refer to two copies of B_N, one above the other with the lower copy shifted j indices to the right and we use $\bar{d}(B_N, S^j B_N)$ to denote the $\bar{d}$ distance measured over the overlap of these two strings.

PROPOSITION 5.12: *Let M and $\epsilon_0 > 0$ be fixed. Then there is a $\delta_0 > 0$ so that for all sufficiently large N, if α and β are two substrings of B_N with $|\alpha| = |\beta| \geq \epsilon_0 |B_N|$ and α appears at i_0 in B_N while β appears at $i_0 + j$, $1 \leq j \leq M$, then*

$$\bar{d}(\alpha, \beta) > \delta_0.$$

Proof: An n-block begins with a single symbol "1" followed by a string of "0"'s of length n, so it is immediate that for fixed $n > M$

$$\delta_n = \inf_{1\leq j\leq M} \bar{d}(B_n, S^j B_n) > 0.$$

Fix $N_0 > M$ large enough so that for all $N > N_0$ at most $\frac{1}{10}\epsilon_0$ of the indices in an N-block are not covered by N_0 blocks inside B_N. We may also assume $M < \frac{1}{10} b_{N_0}$. Now select $N \gg N_0$ (we will specify this more precisely later) and let β be a fixed substring of B_N, $|\beta| > \epsilon_0 b_N$, and let α be the substring below β in $S^j B_N$. Since $N > N_0$ there are at most $\frac{1}{10}|\beta| = \frac{1}{10}|\alpha|$ spacers between N_0 blocks in either α or β (or both). The remaining indices correspond to N_0 block overlaps. Since there are at most $\left[|\beta|/b_{N_0} + 2\right]$ N_0 blocks in B_N meeting β, the number of indices covered by β arising from the overlap of the leftmost part of an N_0 block in β and the rightmost part of an N_0 block in α does not exceed $j\left[|\beta|/b_{N_0} + 1\right] \leq \frac{1}{10}|\beta| + M$. The remaining indices correspond to N_0 block overlaps from shifts by j. Finally, removing the two endmost N_0 block overlaps which may be partial, accounting for at most $2b_{N_0}$ indices we obtain

$$\bar{d}(\beta, \alpha) \geq \frac{1}{|\beta|}\left\{|\beta| - \frac{3}{10}|\beta| - M - 2b_{N_0}\right\}\delta_{N_0} = \left[\frac{7}{10} - \left[\frac{M + 2b_{N_0}}{|\beta|}\right]\right]\delta_{N_0}.$$

Now, if N is sufficiently large, since $|\beta| > \epsilon_0 b_{N_0}$, this quantity will exceed $\delta_{N_0}/2$ which we take for δ_0. ∎

We combine the two previous results as

PROPOSITION 5.13: *Let $\epsilon_0 > 0$ be fixed. Then there exists a $\delta_0 > 0$ and $N_0 < \infty$ so that if $N \geq N_0$ and α and β are two substrings of B_N appearing at i_0 and $i_0 + j$ respectively with $j \geq 1$ and $|\alpha| = |\beta| \geq \epsilon_0 |B_N|$, then*

$$\bar{d}(\alpha, \beta) > \delta_0.$$

Finally, we lift this $\bar{d}$ separation property of n-blocks to full names. The issues are essentially as in Example 4.2) — a modification of the argument there gives

PROPOSITION 5.14: *S satisfies $\bar{d} > \delta_0/20$. (δ_0 from Proposition 5.13.)*

Proof: Again, as in Lemma 4.5, fix $x \neq y \in A^{\mathbb{Z}}$ where we may assume without loss of generality that $x_0 \neq y_0$. Obtain $\delta_0 > 0$ and $N_0 < \infty$ from Proposition 5.13 using $\epsilon_0 = \frac{1}{10}$. Let us also assume that x and y have been chosen from the set of full measure given by property 2.5) and Lemma 2.7 with $\delta_0 = \frac{1}{10}$ so that, for all large enough N

(5.15) x_0 and y_0 are inside N-blocks $B_N(x)$ and $B_N(y)$ respectively (the time-zero N-blocks).

(5.16) The length of the string of spacer symbols in $x_{-\infty}^{\infty}$ (resp. $y_{-\infty}^{\infty}$) immediately to the right of $B_N(x)$ (resp. $B_N(y)$) does not exceed $\frac{1}{10}|B_N|$.

CASE I: For infinitely many N, $B_N(x)$ and $B_N(y)$ are shifted relative to each other by j indices with $\frac{1}{10}|B_N| < j < \frac{8}{10}|B_N|$.

For such N, let $B_N^+(x)$ (resp. $B_N^+(y)$) denote the first N-block to the right of $B_N(x)$ (resp. $B_N(y)$). Evidently $B_N^+(x)$ and $B_N^+(y)$ do not line up and overlap by more than $\frac{1}{10}|B_N|$ units. By Proposition 5.13), the $\bar{d}$ distance measured on this overlap is at least δ_0 and it appears within an interval of $2|B_N|$ indices from the origin so the $\bar{d}$ distance measured over the interval from the origin to the rightmost index of the overlap interval is at least $\frac{1}{20}\delta_0$.

CASE II: For infinitely many N the shift between $B_N(x)$ and $B_N(y)$ is at least $\frac{8}{10}|B_N|$. Then we can replace either $B_N(x)$ with $B_N^+(x)$ or $B_N(y)$ with $B_N^+(y)$ to obtain an interval in $\mathbb{Z}^+$ over which $\bar{d} > \delta_0$. Complete the argument as in Case I.

Otherwise, we have

CASE III: For all large N, $B_N(x)$ and $B_N(y)$ overlap by at least $\frac{9}{10}|B_N|$ and hence, for infinitely many N an interval of at least $\frac{4}{10}|B_N|$ in the time-zero N-block overlap lies in $\mathbb{Z}^+ \cup \{0\}$ (or $\mathbb{Z}^- \cup \{0\}$) and we may apply Proposition 5.13 directly obtaining again, $\bar{d} > \delta_0$ over these intervals.

In all cases we find either

$$\limsup_n \bar{d}(x_0^n, y_0^n) > \frac{1}{20}\delta_0$$

or

$$\limsup_n \bar{d}(x_{-n}^0, y_{-n}^0) > \frac{1}{20}\delta_0$$

and the result has been shown. ∎

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